Rigid Body Kinematics

A rigid body is a system of particles for which the distances between the particles remain unchanged.
Consider an object /body and suppose particles of that body located by position vectors from reference axes attached to and rotating with the body.

In case of rigid body, there will be no change in any position vector as measured from these axes.

This is, however, an ideal situation since all solid materials change shape to some extent when forces are applied to them.

If the movements associated with the changes in shape are very small compared with the movements of the body as a whole, then the assumption of rigidity is acceptable.

In rigid-body kinematics, same relationships are used (as in particle motion) but the equations for rotational motion of the body are also included.

Thus rigid-body kinematics involves both linear and angular displacements, velocities, and accelerations.

The knowledge of kinematics of rigid bodies is used in devices that generate, transmit, or control certain motions such as gears, cams, and several other mechanical parts.

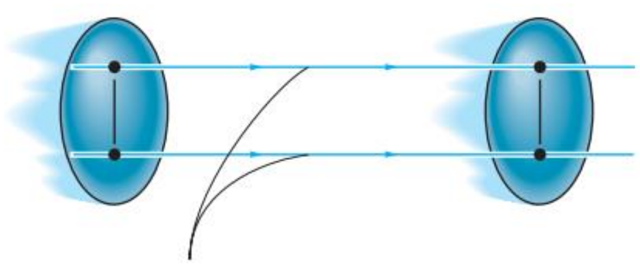
The displacement, velocity, and acceleration data of mechanical parts is required to determine their design geometry.

Plane motion

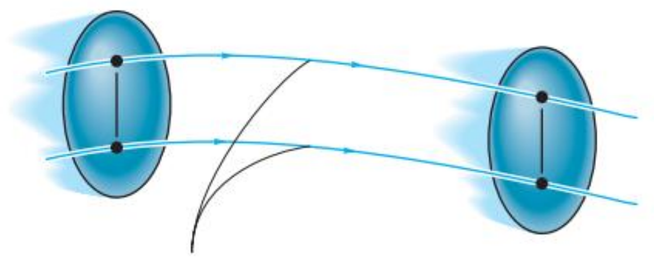
In plane motion, the motion of a rigid body is such that all its points move parallel to some fixed plane.

or, a rigid body executes plane motion when all parts of the body move in parallel planes.

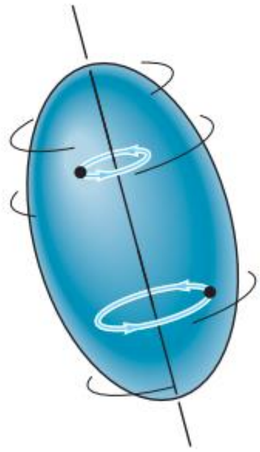
The planar motion occurs when the path of the particles (of a rigid body) is such that their (particles') locations remain equidistant from a reference fixed straight plane.



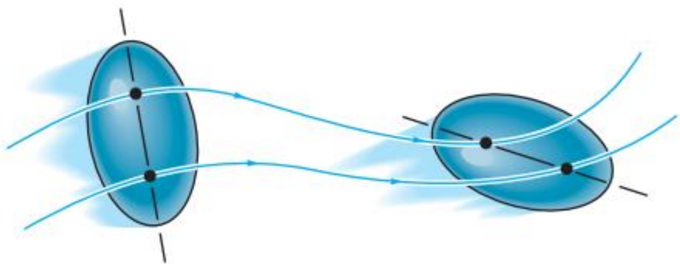
Path of rectilinear translation
(a)



Path of curvilinear translation
(b)



Rotation about a fixed axis
(c)



General plane motion
(d)

For convenience, we consider the plane of motion to be the plane which contains the mass center. The body is treated as a thin slab whose motion is confined to the plane of the slab.

There are three types of rigid-body planar motion. In order of increasing complexity, they are:

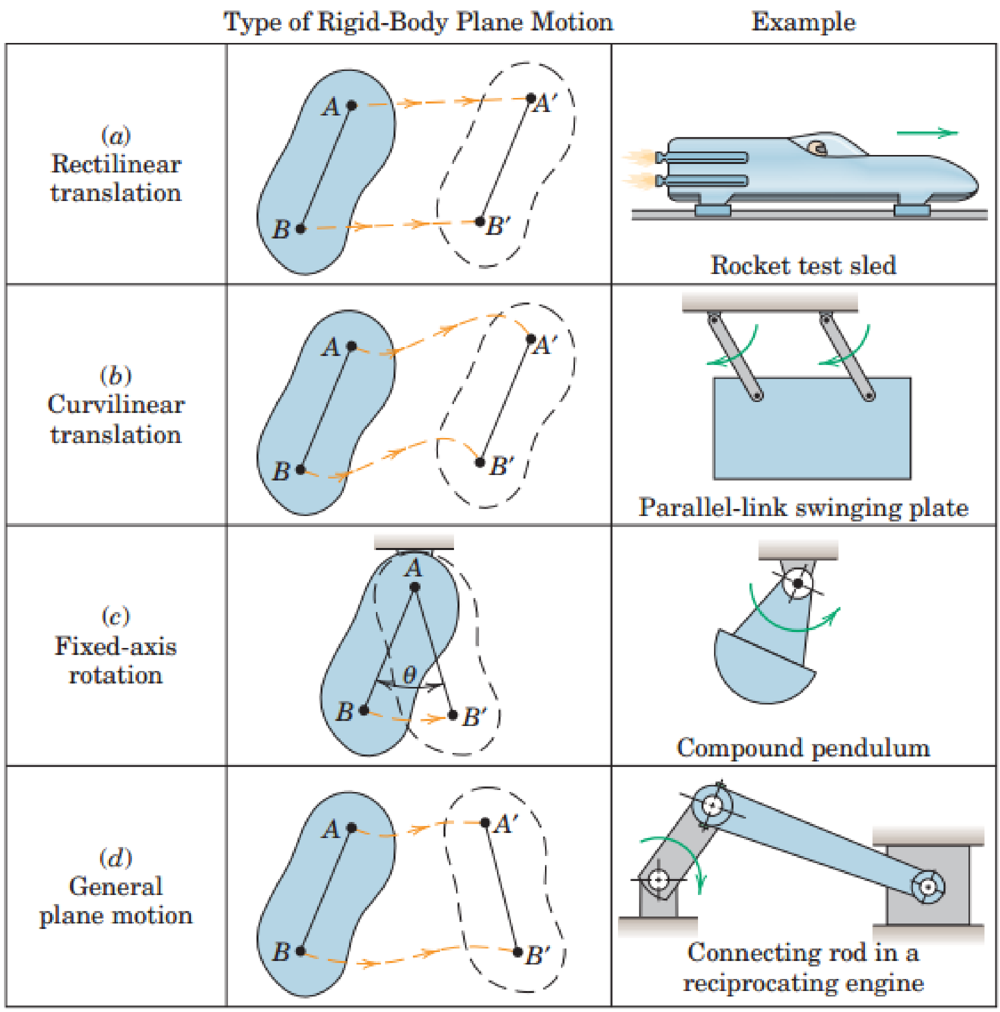
Translation

It is defined as any motion in which every line in the body remains parallel to its original position (orientation) at all times.

In translation there is no rotation of any line in the body.

In rectilinear translation, all points in the body move in parallel straight lines.

In curvilinear translation, the paths of motion are along curved lines and all points move on congruent curves.



Rotation about a fixed axis is the angular motion about the axis.

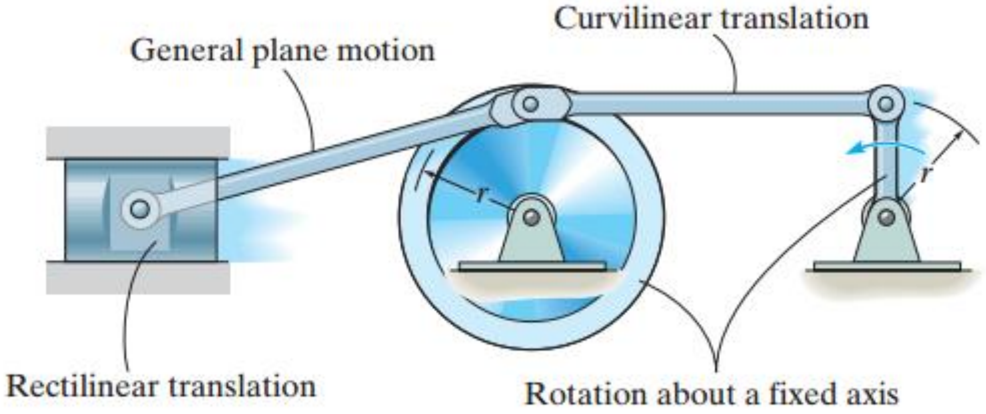
All particles in a rigid body move in circular paths about the axis of rotation, and all lines in the body rotate (except passing through axis)

through the same angle.

General plane motion

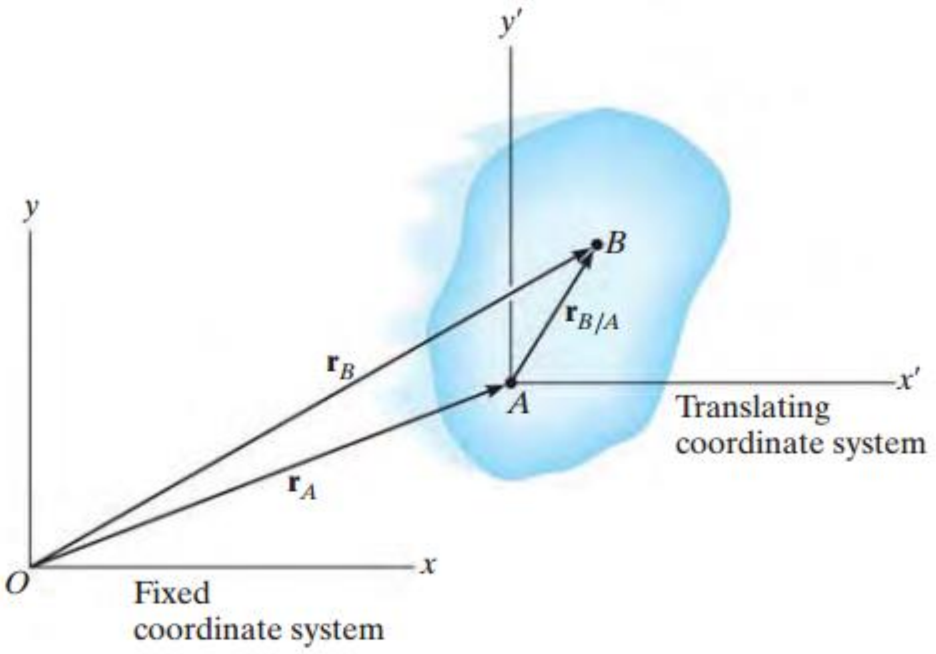
When a body is subjected to general plane motion, it undergoes a combination of translation and rotation.

In Figure, the translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.



Analysis of Translation Motion

Consider a rigid body which is subjected to either rectilinear or curvilinear translation in the x–y plane.



Position: The locations of points A and B on the body are defined with respect to fixed x, y reference frame using position vectors r_A and r_B.

The translating x, y coordinate system is fixed to the body and has its origin at A, hereafter referred to as the base point. The position of B with respect to A is denoted by the relative-position vector $r_{B/A}$.

By vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Velocity A relation between the instantaneous velocities of A and B is obtained by taking the time derivative of this equation, which yields

$$\mathbf{v}_B = \mathbf{v}_A + d\mathbf{r}_{B/A}/dt.$$

The term

$$d\mathbf{r}_{B/A}/dt = \mathbf{0}$$

since the magnitude of $r_{B/A}$ is constant by definition of a rigid body, and because the body is translating the direction of $r_{B/A}$ is also constant. Therefore,

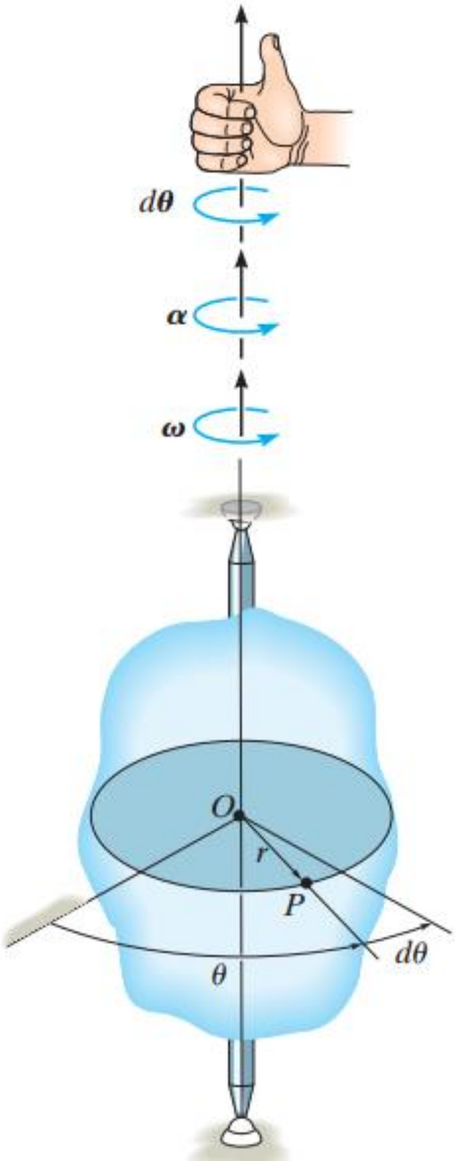
$$\mathbf{v}_B = \mathbf{v}_A \qquad \mathbf{a}_B = \mathbf{a}_A$$

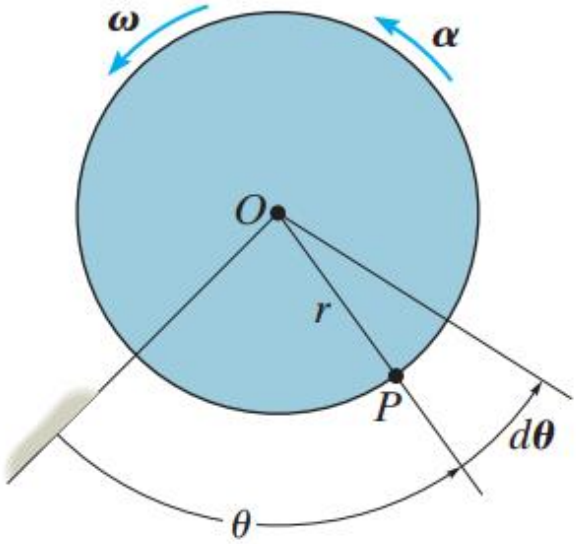
The above two equations indicate that all points in a rigid body subjected to either rectilinear or curvilinear translation move with the same velocity and acceleration.

Therefore, the results of kinematic analysis of particle motion can also be used to specify the kinematics of points located in a translating rigid body.

<u>Analysis for Rotation about a Fixed Axis</u>

When a body rotates about a fixed axis, any point P located in the body travels along a circular path.





<u>Angular Motion</u>

Since a point is without dimension, it cannot have angular motion (rotation). Only lines or bodies undergo angular motion.

For example, consider the body shown and the angular motion of a radial line r located within the shaded plane.

<u>Angular Position</u>

At the instant shown, the angular position of r is defined by the angle θ, measured from a fixed reference line.

Angular Displacement

The change in the angular position, which is measured as a differential angle dθ, is called the angular displacement.

The magnitude of dθ is measured in degrees, radians, or revolutions, where 1 rev = 2π rad.

Since motion is about a fixed axis, the direction (or line of action) of dθ may be expressed by the vector normal to the plane of rotation with a sense governed by the right-hand rule.

That is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or dθ, points upward.

In two dimensions, as shown by the top view of the shaded plane, both θ and dθ are counterclockwise, and so the thumb points outward from the page.

Angular Velocity

The time rate of change in the angular position is called the angular velocity ω (omega). Since dθ occurs during an instant of time dt, then,

$$(\circlearrowleft +) \qquad \boxed{\omega = \frac{d\theta}{dt}} \qquad (1)$$

The magnitude of this vector is often measured in rad/s.

ω can be expressed here in scalar form since its direction is already known (considered to be along the axis of rotation).

When indicating the angular motion we can have the rotation as clockwise or counterclockwise.

The counterclockwise rotation is arbitrarily chosen as positive and indicated this by the curl shown in parentheses next to Equation (1).

Angular Acceleration

The angular acceleration α (alpha) measures the time rate of change of the angular velocity. The magnitude of this vector is

$$(\circlearrowleft +) \qquad \boxed{\alpha = \frac{d\omega}{dt}} \qquad \text{or} \qquad \alpha = \frac{d^2\theta}{dt^2} \qquad (2)$$

The line of action of α is the same as that for ω; however, its sense of direction depends on whether ω is increasing or decreasing.

If ω is decreasing, then α is called an angular deceleration and therefore has a sense of direction which is opposite to ω.

By eliminating dt from Equations (1) and (2), a differential relation between the angular acceleration, angular velocity, and angular displacement is obtained

$$\boxed{\alpha \, d\theta = \omega \, d\omega} \qquad (3)$$

Constant Angular Acceleration

If the angular acceleration of the body is constant i.e. $\alpha = \alpha_c$, then Equations (1), (2) and (3), when integrated, yield a set of formulas which relate the body's angular velocity, angular position, and time.

$$\omega = \omega_0 + \alpha_c t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

Constant Angular Acceleration

Here θ_0 and ω_0 are the initial values of the body's angular position and angular velocity, respectively.

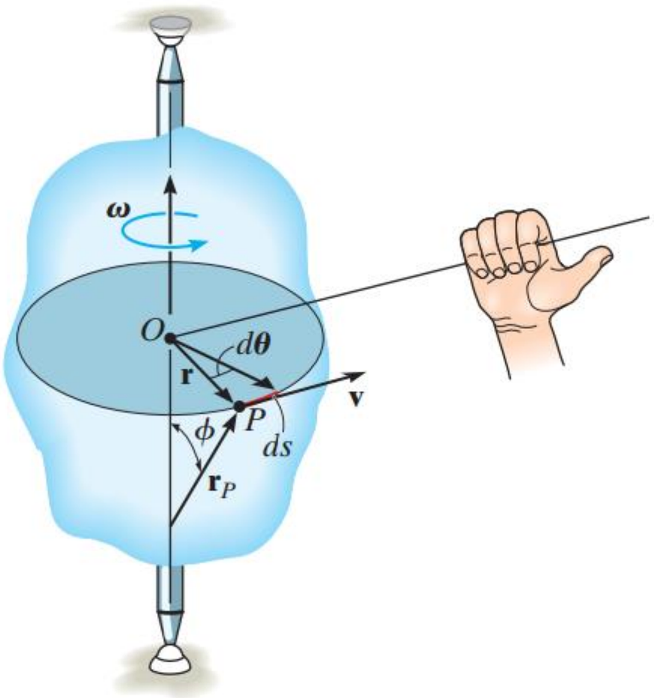
Position and Displacement

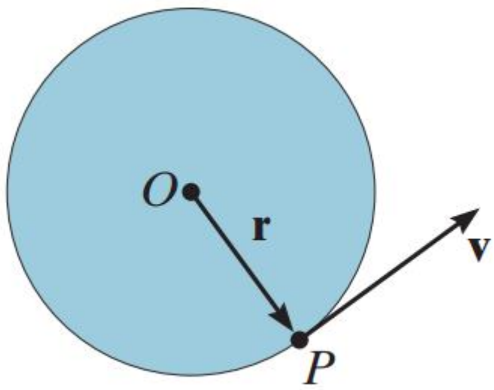
The position of P is defined by the position vector **r**, which extends from O to P. If the body rotates dθ then P will displace ds = rdθ.

Velocity

The magnitude of velocity of P can be found by dividing ds = rdθ by dt to get following equation:

$$\boxed{v = \omega r}$$





The direction of v is tangent to the circular path.

The magnitude and direction of v at point P can also be determined from any point on the axis of rotation. Let, r_P is a position vector directed to point P.

v can now be determined by taking the cross product of ω and r_P

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P} \quad (4)$$

The direction of v is established by the right-hand rule.

The fingers of the right hand are curled from ω toward r_P (v “cross” r_P). The thumb indicates the direction of v, which is tangent to the path in the direction of motion.

The magnitude of v in Eq. 4 is

$$v = \omega r_P \sin\phi$$

In addition from Figure

$$r = r_P \sin\phi,$$

Thus

$$v = \omega r.$$

<u>Acceleration</u>

The acceleration of P can be expressed in terms of its normal and tangential components.

$$a_t = dv/dt \text{ and } a_n = v^2/\rho$$

where $\rho = r$, $v = \omega r$, and $\alpha = d\omega/dt$,

$$\boxed{a_t = \alpha r} \qquad \boxed{a_n = \omega^2 r}$$

The tangential component of acceleration represents the time rate of change in the velocity's magnitude.

If the speed of P is increasing, then a_t acts in the same direction as v. If the speed is decreasing, then a_t acts in the opposite direction as v.

The normal component of acceleration represents the time rate of change in the velocity's direction. The direction of a_n is always toward O, the center of the circular path,

Like the velocity, the acceleration of point P can be expressed in terms of the vector cross product. Taking the time derivative of

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt}$$

but $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ and $d\mathbf{r}_P/dt = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$.

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P)$$

From the definition of the cross product, the first term on the right has a magnitude

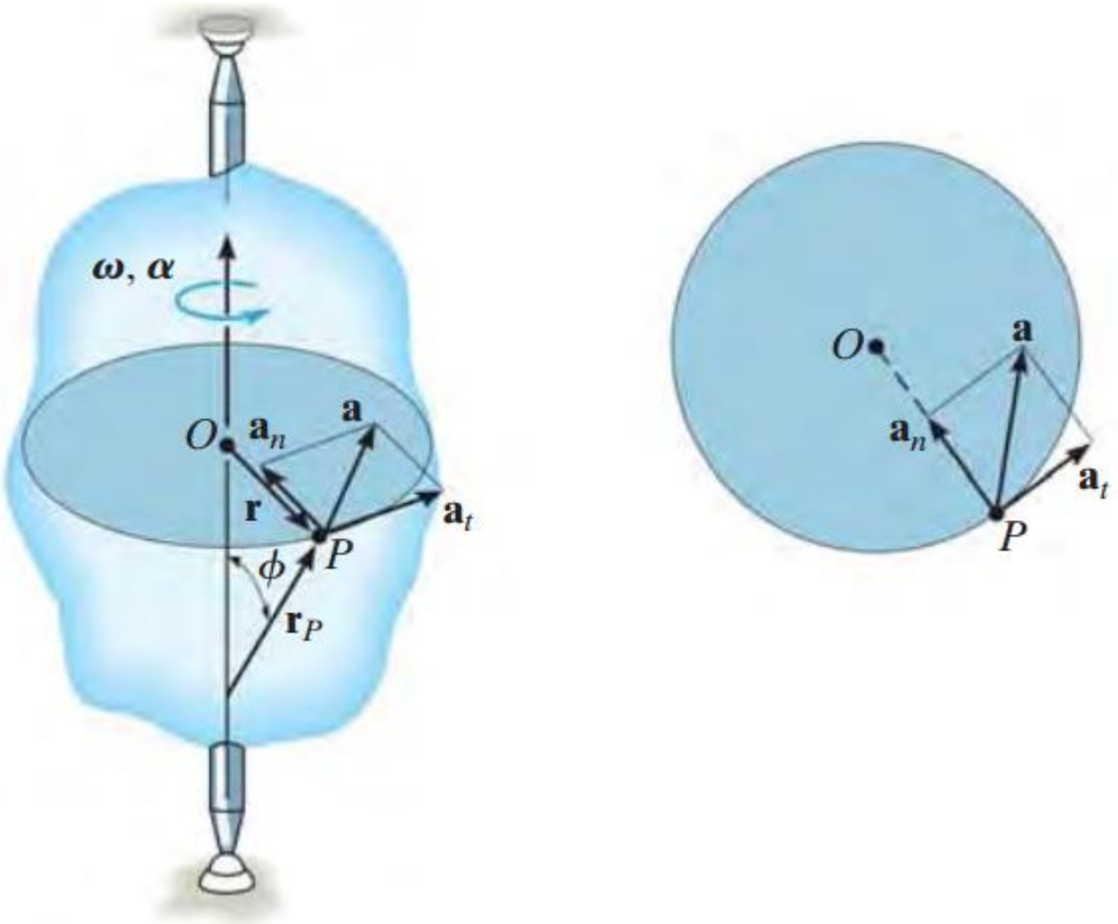
$$a_t = \alpha r_P \sin\phi = \alpha r$$

Using the right-hand rule, $\alpha \times r_P$ is in the direction of a_t,

Likewise, the second term has a magnitude

$$a_n = \omega^2 r_P \sin\phi = \omega^2 r$$

Applying the right-hand rule twice, first to determine the result $\mathbf{v} = \omega \times \mathbf{r}_P$ then $\omega \times v$, it can be seen that this is in the same direction as a_n, shown in Figure.



Noting that this is also the same direction as -r, which lies in the plane of motion, we can express a_n in a much simpler form as $\mathbf{a}_n = -\omega^2\mathbf{r}$.

$$\boxed{\begin{aligned} \mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r} \end{aligned}}$$

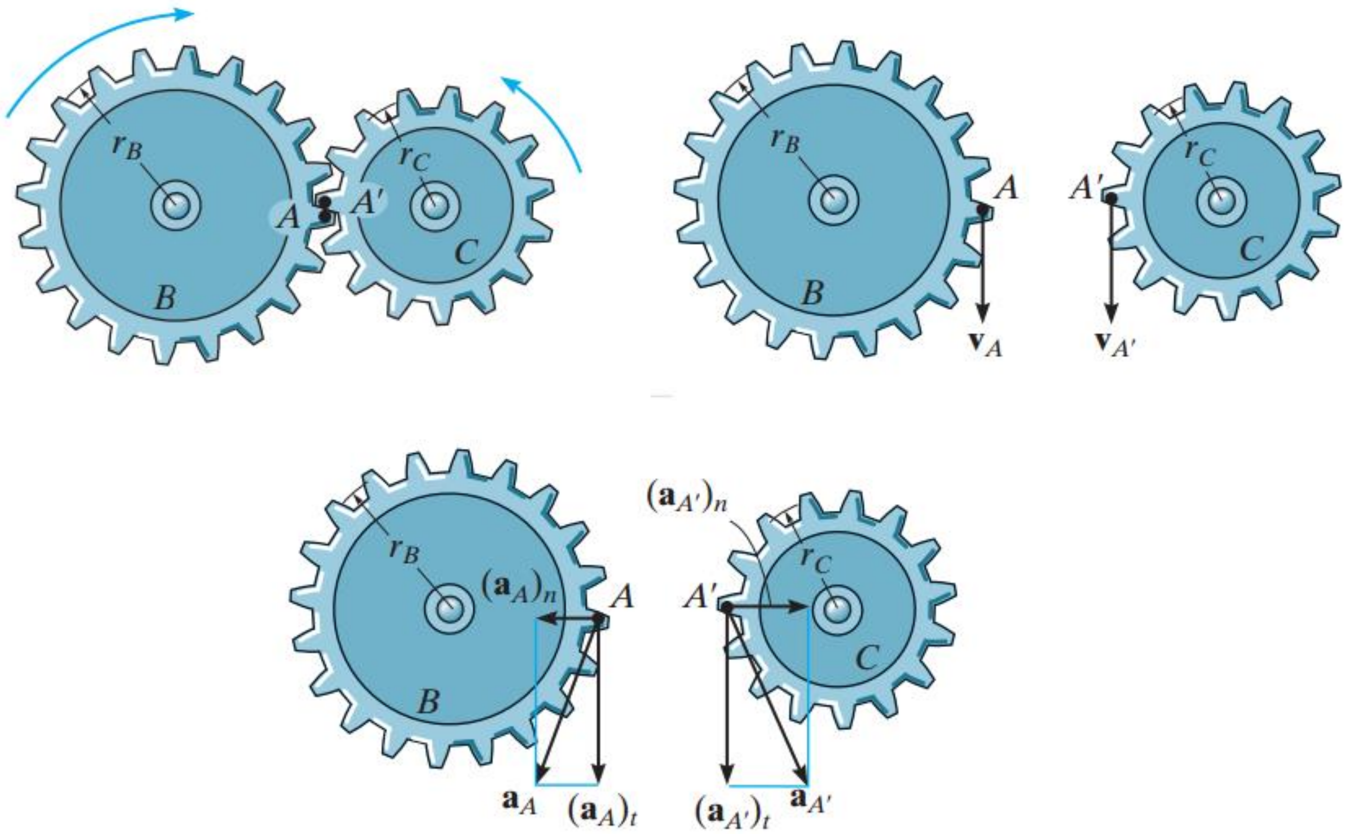
If two rotating bodies contact one another the velocity and tangential components of acceleration of the points in contact will be the same.

However, the normal components of acceleration will not be the same.

For example, consider the two meshed gears in Figure. Point A is located on gear B and a coincident point A’ is located on gear C.

Due to the rotational motion, $v_A = v_{A'}$, and as a result, $\omega_B r_B = \omega_C r_C$ or $\omega_B = \omega_C(r_C/r_B)$. In the similar manner, $(a_A)_t = (a_{A'})_t$, so that $\alpha_B = \alpha_C(r_C/r_B)$;

Since both points follow different circular paths, $(a_A)_n \neq (a_{A'})_n$ and therefore, $a_A \neq a_{A'}$



Absolute Motion Analysis (General Planar motion)

In the earlier discussion of absolute analysis (particle motion), no angular quantities were considered in the geometric relations.

While dealing with rigid-body motion, however, the relations include both linear and angular variables.

Similarly, the time derivatives of these quantities involve both linear and angular velocities and linear and angular accelerations.

A body subjected to general plane motion undergoes a simultaneous translation and rotation.

The motion can be completely specified by knowing both (i) the angular rotation of a line fixed in the body and (ii) the motion of a point on the body.

One way to relate these motions is to use a rectilinear position coordinate s to locate the point along its path and an angular position coordinate θ to specify the orientation of the line.

The two coordinates are then related using the geometry of the problem.

If the geometric configuration is complex, analysis by the principles of relative motion is preferable.

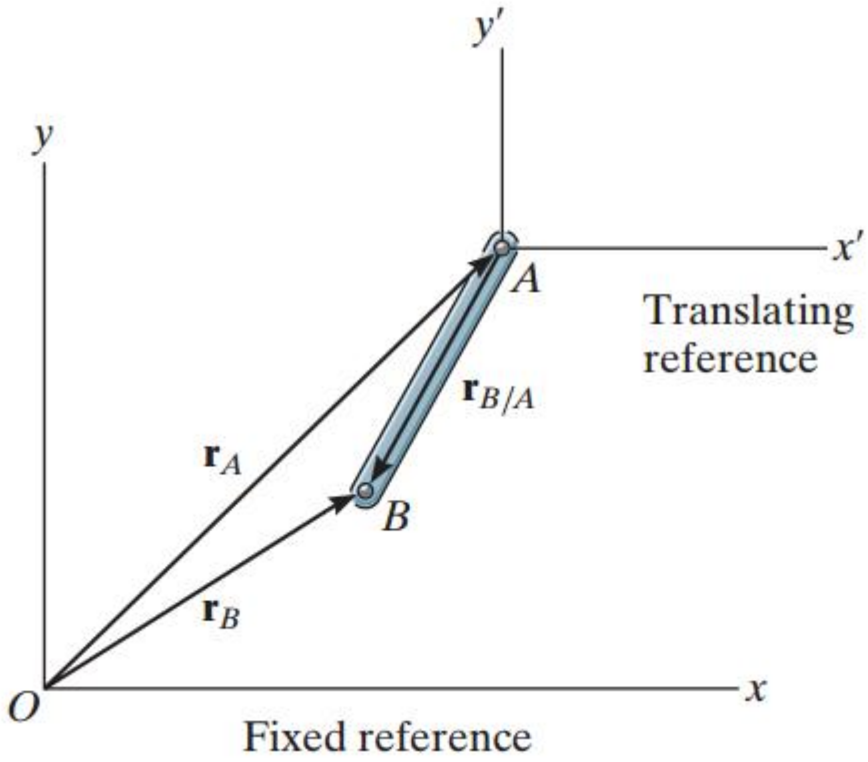
Relative-Motion Analysis: Velocity

The general plane motion of a rigid body can be described as a combination of translation and rotation.

To view these "component" motions separately, a relative-motion analysis involving two sets of coordinate axes is performed.

The x, y coordinate system is fixed and measures the absolute position of two points A and B on the body (here represented as a bar in Figure).

The origin of the x', y' coordinate system will be attached to the selected "base point" A, which generally has a known motion.



The axes of this coordinate system translate with respect to the fixed frame but do not rotate with the bar

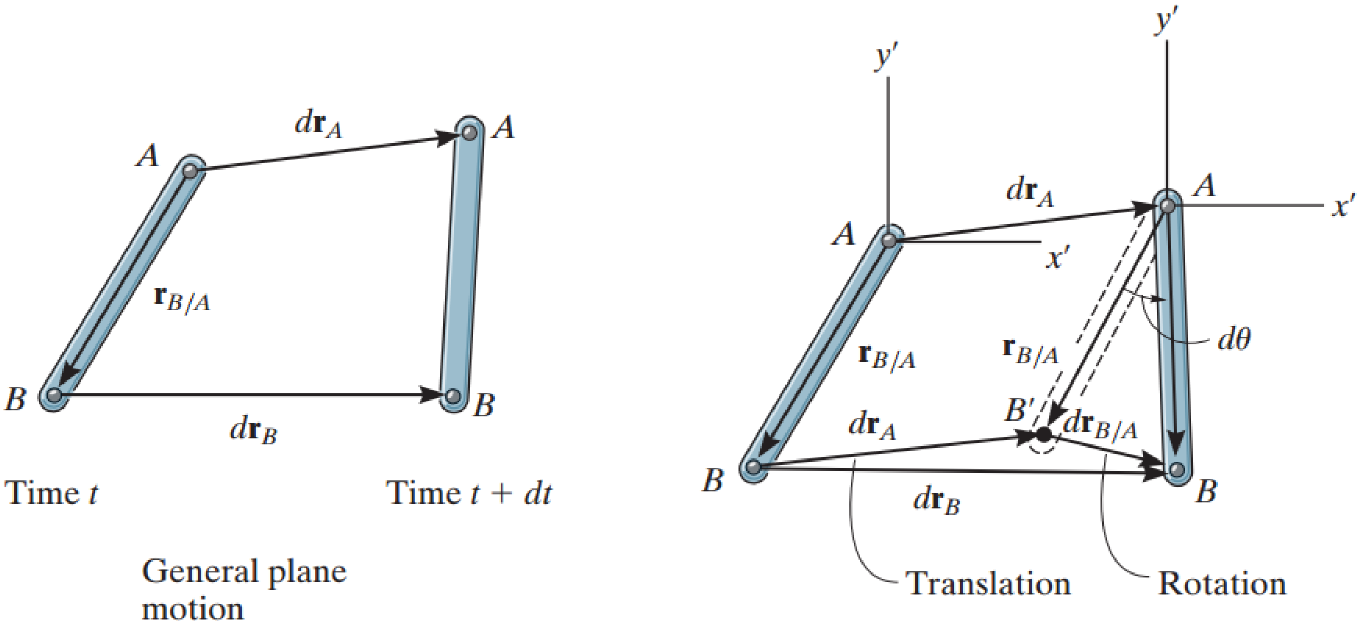
Position: The position vector r_A in Figure specifies the location of the "base point" A, and the relative-position vector $r_{B/A}$ locates point B with respect to point A. By vector addition, the position of B is then

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Displacement: During an instant of time dt, points A and B undergo displacements dr_A and dr_B. If the general plane motion is considered by its component parts then:

(i) The entire bar first translates by an amount dr_A so that A, the base point, moves to its final position and point B moves to B'.

(ii) The bar is then rotated about A by an amount $d\theta$ so that B' undergoes a relative displacement $dr_{B/A}$ and thus moves to its final position B.



Due to the rotation about A, $dr_{B/A} = r_{B/A}\, d\theta$, and the displacement of B is

$$\underbrace{d\mathbf{r}_B}_{\text{due to translation and rotation}} = \underbrace{d\mathbf{r}_A}_{\text{due to translation of } A} + \underbrace{d\mathbf{r}_{B/A}}_{\text{due to rotation about } A}$$

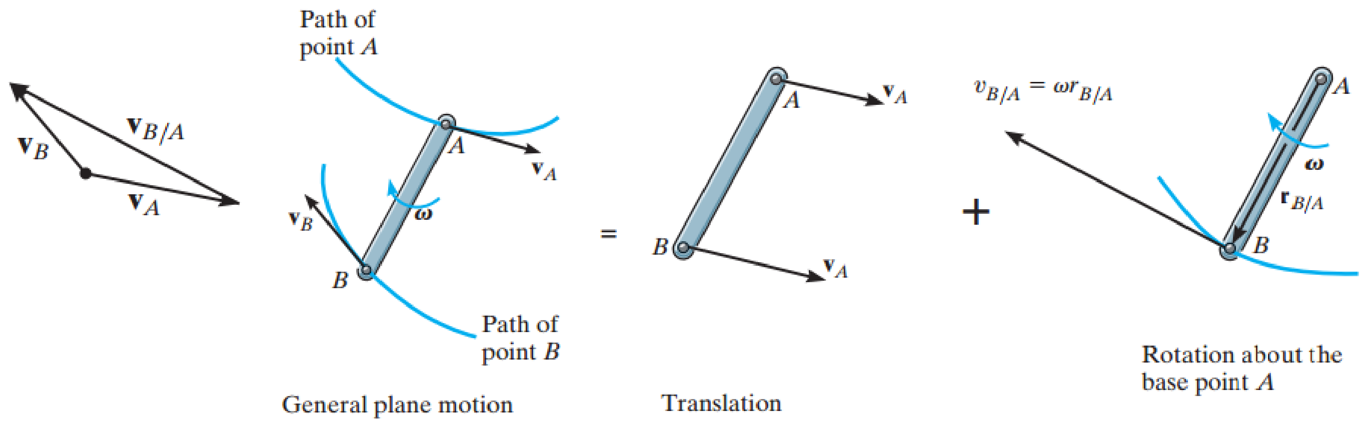
<u>Velocity</u>: To determine the relation between the velocities of points A and B, the time derivative of the position equation is taken, or simply the displacement equation can be divided by dt.

$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt}$$

The terms $dr_B/dt = v_B$ and $dr_A/dt = v_A$ are measured with respect to the fixed x, y axes and represent the absolute velocities of points A and B, respectively.

Since the relative displacement is caused by a rotation, the magnitude of the third term is:

$$dr_{B/A}/dt = r_{B/A}\, d\theta/dt = r_{B/A}\dot{\theta} = r_{B/A}\omega$$



The term $r_{B/A}\,\omega$ or $v_{B/A}$ is relative velocity, since it represents the velocity of B with respect to A as measured by an observer fixed to the translating x, y axes.

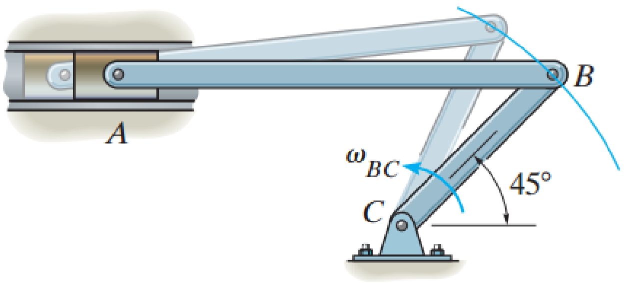
In other words, the bar appears to rotate an angular velocity ω about the z' axis passing through A.
Consequently, $v_{B/A}$ has a magnitude of $v_{B/A} = \omega r_{B/A}$ and a direction which is perpendicular to $r_{B/A}$. We therefore have

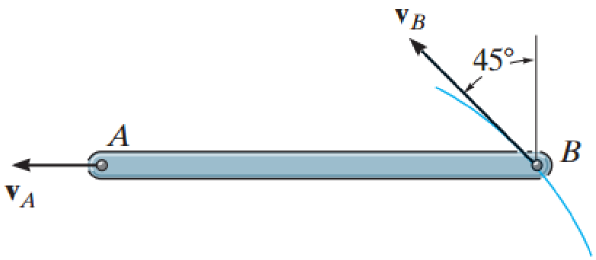
$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}}$$

The equation $v_B = v_A + v_{B/A}$ states is that the velocity of B is determined by considering the entire bar to translate with a velocity of v_A, and rotate about A with an angular velocity ω,

Since the relative velocity $v_{B/A}$ represents the effect of circular motion, about A, this term can be expressed by the cross product

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}}$$





The velocity equation may be used in a practical manner to study the general plane motion of a rigid body which is either pin connected to or in contact with other moving bodies.

When applying this equation, points A and B should generally be selected as points on the body which are pin-connected.

<u>Relative-Motion Analysis: Acceleration</u>

An equation that relates the accelerations of two points on a bar (rigid body) subjected to general plane motion may be determined by differentiating $v_B = v_A + v_{B/A}$ with respect to time.

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

The terms $dv_B/dt = a_B$ and $dv_A/dt = a_A$ are measured with respect to a set of fixed x, y axes and represent the absolute accelerations of points B and A.

The last term represents the acceleration of B with respect to A as measured by an observer fixed to translating x', y' axes which have their origin at the base point A.

$a_{B/A}$ can be expressed in terms of its tangential and normal components;

$$\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

From previous results

$$(a_{B/A})_t = \alpha r_{B/A} \text{ and } (a_{B/A})_n = \omega^2 r_{B/A}$$

The relative-acceleration equation can be written in the form

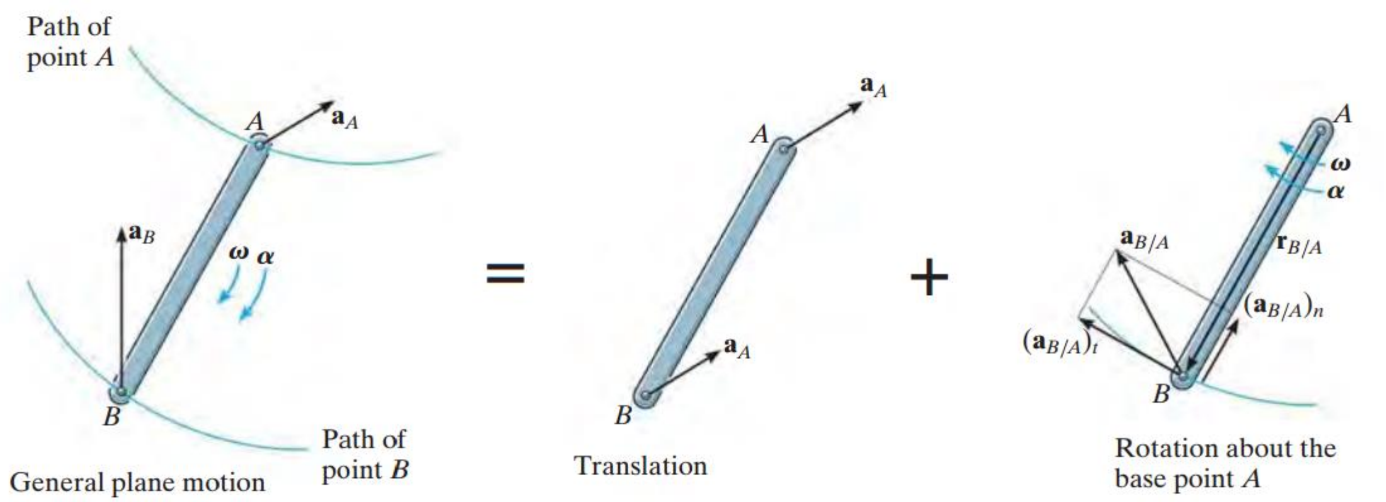
$$\boxed{\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n}$$

The terms in above equation are represented graphically in Figure.

It is seen that at a given instant the acceleration of B, is determined by considering the bar to

(i) translate with an acceleration a_A and simultaneously

(ii) rotate about the base point A with an instantaneous angular velocity ω and angular acceleration α.



Since the relative-acceleration components represent the effect of circular motion observed from translating axes having their origin at the base point A, these terms can be expressed as

$$(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} \qquad (\mathbf{a}_{B/A})_n = -\omega^2 \mathbf{r}_{B/A}$$

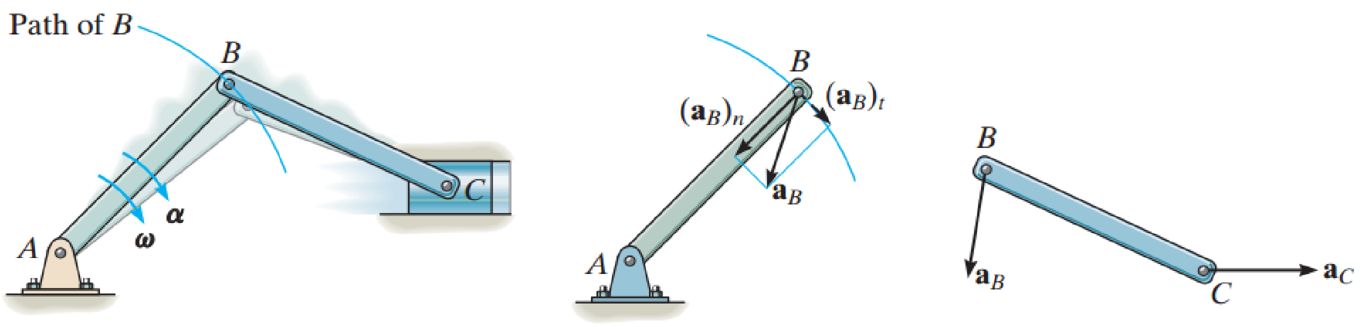
The absolute acceleration of B now is written as:

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

The above equation can be applied to study the accelerated motion of a rigid body which is pin connected as shown in Figure.

The points which are coincident at the pin move with the same acceleration, since the path of motion over which they travel is the same.

For example, point B lying on either rod BA or BC of the crank mechanism shown in Figure has the same acceleration, since the rods are pin connected at B.



Here the motion of B is along a circular path, so that a_B can be expressed in terms of its tangential and normal components.

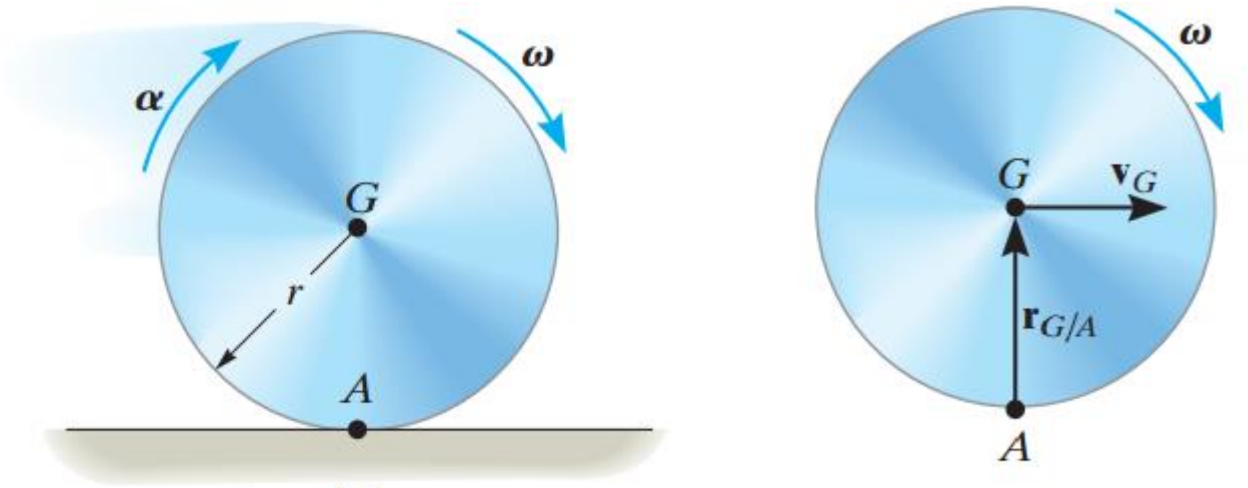
At the other end of rod BC point C moves along a straight-lined path, which is defined by the piston. Hence, a_C is horizontal.

Consider a disk that rolls without slipping as shown in Figure.

As a result, v_A (or v_{IC}) = 0 and the velocity of the mass center G is

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \mathbf{0} + (-\omega\mathbf{k}) \times (r\mathbf{j}) \quad \text{or} \quad v_G = \omega r$$

Since G moves along a straight line, its acceleration in this case can be determined from the time derivative of its velocity



(a)

$$\frac{dv_G}{dt} = \frac{d\omega}{dt} r \qquad a_G = \alpha r$$

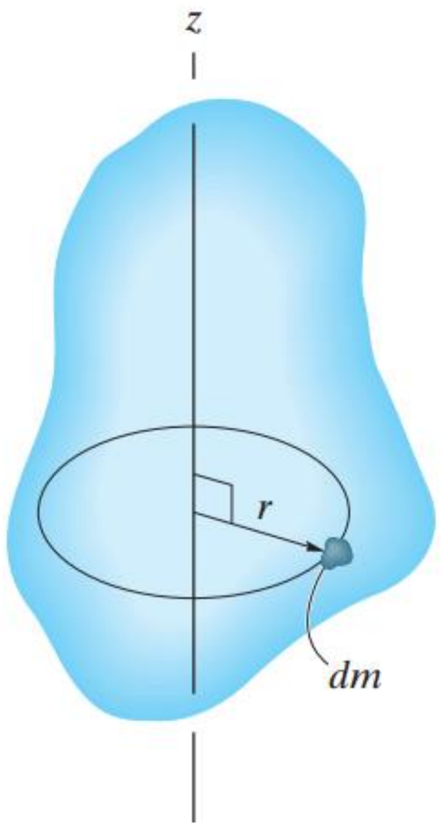
<u>Mass Moment of Inertia</u>

Since a body has a definite size and shape, an applied non-concurrent force system can cause the body to both translate and rotate.

The rotational aspects, caused by a moment M, are governed by an equation of the form M = Iα. The symbol I in this equation is termed the mass moment of inertia.

By comparison, the moment of inertia is a measure of the resistance of a body to angular acceleration (M = Iα) in the same way that mass is a measure of the body's resistance to acceleration (F=ma)

If a component has a large moment of inertia about its axis of rotation, once set into motion, it is difficult to stop it. An example is a flywheel on an engine.



$$\boxed{I = \int_m r^2 \, dm}$$

The moment of inertia is defined as the integral of the “second moment” about an axis of all the elements of mass dm which compose the body.

The “moment arm” r is the perpendicular distance from the z axis to the arbitrary element dm.

As the formulation involves r, the value of I is different for each axis about which it is computed.

The axis chosen for analysis generally passes through the body’s mass center G.

Since r is squared in equation, the mass moment of inertia is always a positive quantity.

Common units used for its measurement are $kg \cdot m^2$ or $slug \cdot ft^2$.

If the body consists of material having a variable density, ρ= ρ (x,y,z), the elemental mass dm of the body can be expressed in terms of its density and volume as dm = ρ dV.

$$I = \int_V r^2 \rho \, dV$$

<u>Parallel-Axis Theorem</u> If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other parallel axis can be determined by using the parallel-axis theorem.

Consider a mass m as shown in Figure. Selecting the differential element of mass dm, we can express the moment of inertia of the body about the x axis as:

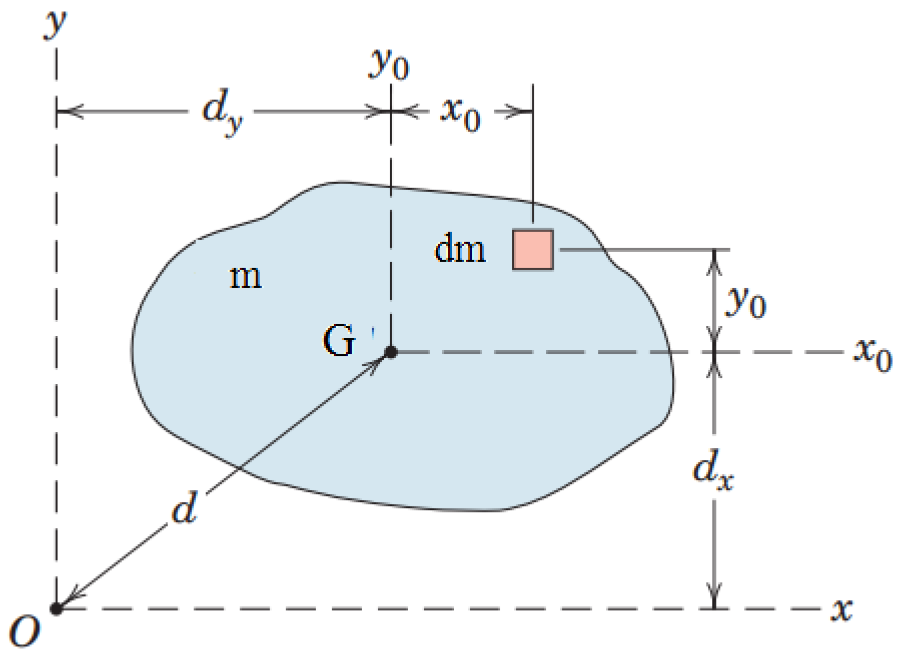
$$\mathrm{dI_x = (y_0 + d_x)^2 dm}$$

$$\mathrm{I_x} = \int \mathrm{y_0^2 dm} + 2\mathrm{d_x} \int \mathrm{y_0 dm} + \mathrm{d_x^2} \int \mathrm{dm}$$

The second integral is zero, since integral of y_0 dm = $\overline{y_0}\mathrm{m}$

$\overline{y_0}$ is zero with the mass center is on the x_0-axis.

$$\mathrm{I_x = I_G + md_x^2}$$



The transfer of axes cannot be made unless one axis passes through the center of mass and unless the axes are parallel.

Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the radius of gyration, k.

This is a geometrical property which has units of length. When it and the body's mass m are known, the body's moment of inertia is determined from the equation:

$$\boxed{I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}}$$

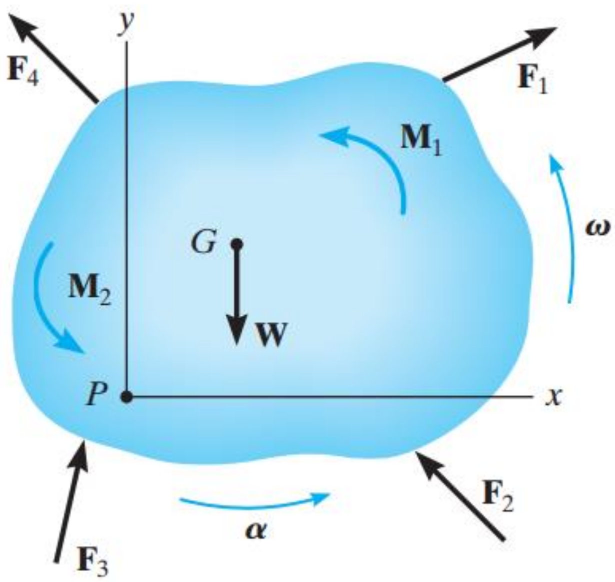
Kinetics of a Rigid Body

Planar Kinetic Equations of Motion

In the study of planar kinetics, the rigid bodies along with their loadings are considered to be symmetrical with respect to a fixed reference plane.

The motion of the body in such analysis is viewed within the reference plane and all the forces (and moments) acting on the body is projected onto the same plane.

An example of an arbitrary body of this type is shown in Figure.



Force Equation

The force equation (which may be referred to as the translational equation of motion) for the mass center of a rigid body states that

'the sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center G'.

For motion of the body in the x-y plane, the force equation of motion may be written in the form of two independent scalar equations,

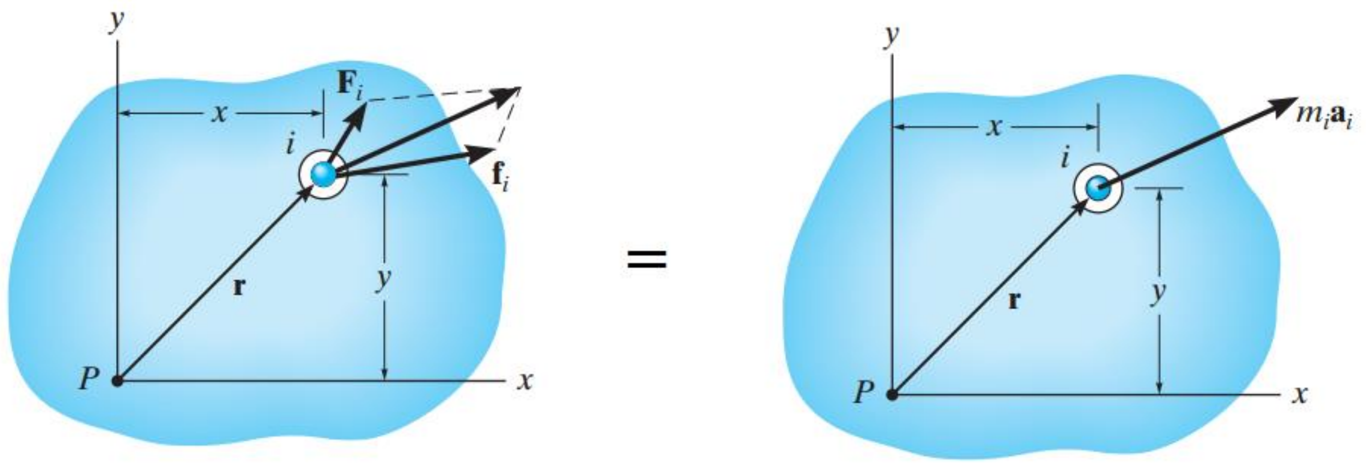
$$\Sigma \mathbf{F} = m\mathbf{a}_G \qquad \Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

<u>Moment Equation</u>

The moment equation determined the effects caused by the moments of the external force system. It is computed about an axis perpendicular to the plane of motion (the z axis) and passing through point P (in Figure).

This equation is also termed as Equation of Rotational Motion.



Particle free-body diagram

Particle kinetic diagram

As shown on the free-body diagram of the ith particle, F_i represents the resultant external force acting on the particle and f_i is the resultant of the internal forces caused by interactions with adjacent particles.

The mass of particle is m_i and force causes acceleration a_i.

Summing moments about point P, we have

$$\mathbf{r} \times \mathbf{F}_i + \mathbf{r} \times \mathbf{f}_i = \mathbf{r} \times m_i \mathbf{a}_i$$

or $$(\mathbf{M}_P)_i = \mathbf{r} \times m_i \mathbf{a}_i$$

The moments about P can also be expressed in terms of the acceleration of point P. Absolute acceleration at location ‘i' can be related with acceleration at P as:

$\mathbf{a}_i = \mathbf{a}_p + \mathbf{a}_{i/p}$ where $a_{i/p} = \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}$

where α is angular acceleration α and ω is angular velocity.

$$
\begin{aligned}
(\mathbf{M}_P)_i &= m_i\mathbf{r} \times (\mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}) \\
&= m_i[\mathbf{r} \times \mathbf{a}_P + \mathbf{r} \times (\boldsymbol{\alpha} \times \mathbf{r}) - \omega^2(\mathbf{r} \times \mathbf{r})]
\end{aligned}
$$

The last term is zero, since $\mathbf{r} \times \mathbf{r} = 0$.

Expressing the vectors with Cartesian components and carrying out the cross-product operations yields:

$$
\begin{aligned}
(M_P)_i\mathbf{k} = m_i\{&(x\mathbf{i} + y\mathbf{j}) \times [(a_P)_x\mathbf{i} + (a_P)_y\mathbf{j}] \\
&+ (x\mathbf{i} + y\mathbf{j}) \times [\alpha\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})]\}
\end{aligned}
$$

$$
\begin{aligned}
(M_P)_i\mathbf{k} &= m_i[-y(a_P)_x + x(a_P)_y + \alpha x^2 + \alpha y^2]\mathbf{k} \\
\circlearrowleft (M_P)_i &= m_i[-y(a_P)_x + x(a_P)_y + \alpha r^2]
\end{aligned}
$$

Letting $m_i \rightarrow dm$ and integrating with respect to the entire mass m of the body, the resultant moment equation is obtained

$$
\circlearrowleft \Sigma M_P = -\left(\int_m y\, dm\right)(a_P)_x + \left(\int_m x\, dm\right)(a_P)_y + \left(\int_m r^2 dm\right)\alpha
$$

Here M_P represents only the moment of the external forces acting on the body about point P. The resultant moment of the internal forces is zero,

The integrals in the first and second terms on the right are used to locate the body's center of mass G with respect to P. The three terms are simplified as

$$\bar{y}m = \int y\, dm \qquad \bar{x}m = \int x\, dm \qquad I_P = \int r^2 dm$$

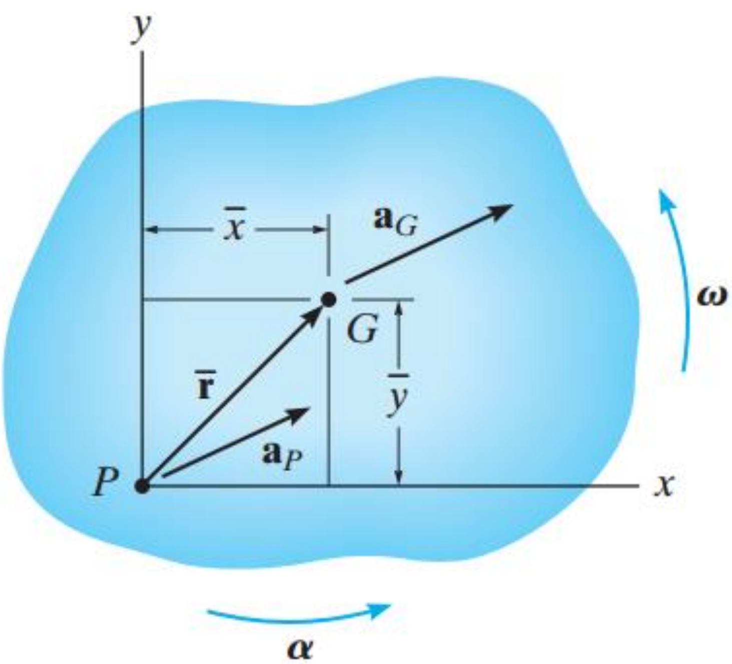
The first two integrals are first moment. $\bar{x}$ is the mean *x* distance of all particles with P i.e distance between G and P.

The product of various masses and distances measured from any reference location gives product of average distance and total mass.

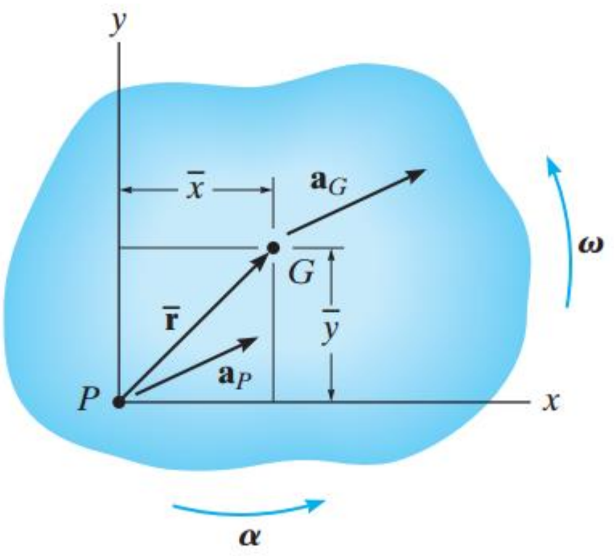
The last integral represents the body's moment of inertia about the z axis

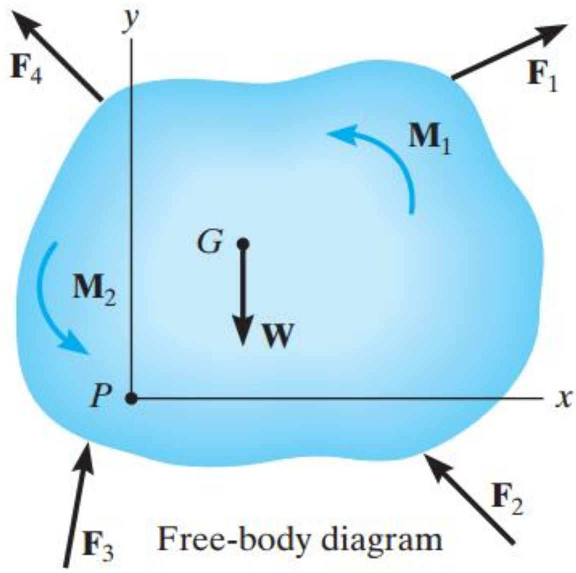
The equation now becomes

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_P)_x + \bar{x}m(a_P)_y + I_P\alpha \qquad (1)$$



It is possible to reduce this equation to a simpler form if point P coincides with the mass center G for the body. If this is the case, then $\bar{x} = \bar{y} = 0$. The equation now becomes $\sum M_G = I_G\alpha$





The moment equation (or rotational equation of motion) states that

"the sum of the moments of all the external forces about the body's mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular acceleration'.

Equation (1) can also be rewritten in terms of the x and y components of a_G and the body's moment of inertia I_G. If point P is located at $(\bar{x}, \bar{y})$, from G then by the parallel-axis theorem,

$I_P = I_G + m(\bar{x}^2 + \bar{y}^2)$ (2)

a_P can be expressed in terms of a_G as

$$\mathbf{a}_G = \mathbf{a}_P + \boldsymbol{\alpha} \times \bar{\mathbf{r}} - \omega^2 \bar{\mathbf{r}}$$

$$(a_G)_x\mathbf{i} + (a_G)_y\mathbf{j} = (a_P)_x\mathbf{i} + (a_P)_y\mathbf{j} + \alpha\mathbf{k} \times (\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) - \omega^2(\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) \quad (3)$$

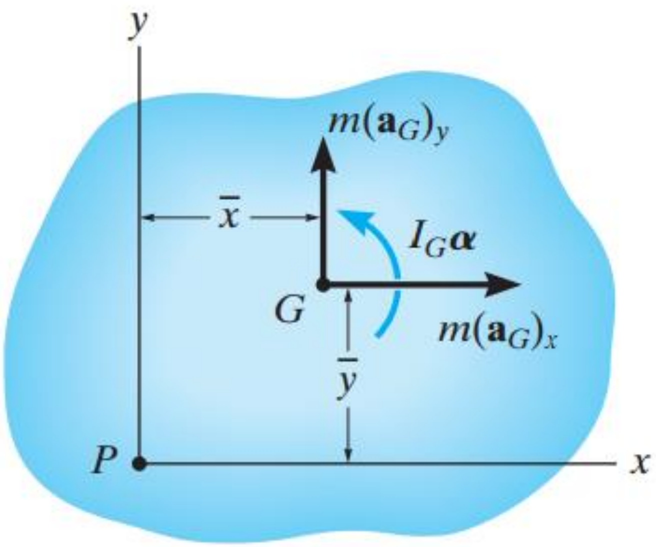
Solving (1), (2) and (3) and rearranging terms gives

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_G)_x + \bar{x}m(a_G)_y + I_G\alpha$$

The result indicates that the sum of moments of the external forces about point P is equal to the sum of the "kinetic moments $(M_k)_P$" i.e.

$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P$$

The kinetic moments include moment of ma_G about P and the moment $I_G\alpha$.



Kinetic diagram

The independent scalar equations thus describing the general plane motion of a symmetrical rigid body are:

$$\Sigma F_x = m(a_G)_x \qquad \Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = I_G\alpha \qquad \Sigma M_P = \Sigma(\mathcal{M}_k)_P$$

Equations of Motion: Translation

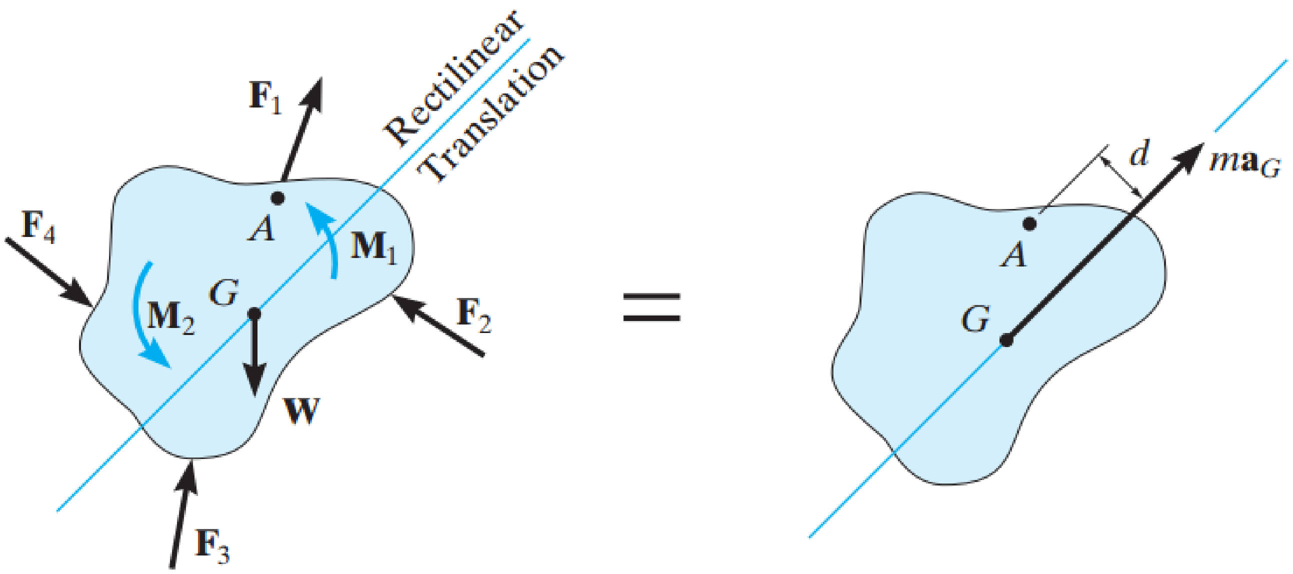
When the rigid body undergoes a translation, all the particles of the body have the same acceleration.

Furthermore, $\alpha = 0$, in which case the moment equation of motion applied at point G reduces to a simplified form, namely, $M_G = 0$.

Rectilinear Translation

The free-body and kinetic diagrams are shown in Figure. The equations of motion are:

$$\Sigma F_x = m(a_G)_x \qquad \Sigma F_y = m(a_G)_y \qquad \Sigma M_G = 0$$



It is also possible to sum moments about other points on or off the body, in which case the moment of ma_G must be taken into account.

For example, if point A is chosen, which lies at a perpendicular distance d from the line of action of ma_G, the following moment equation applies:

$$\circlearrowleft + \Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad \Sigma M_A = (ma_G)d$$

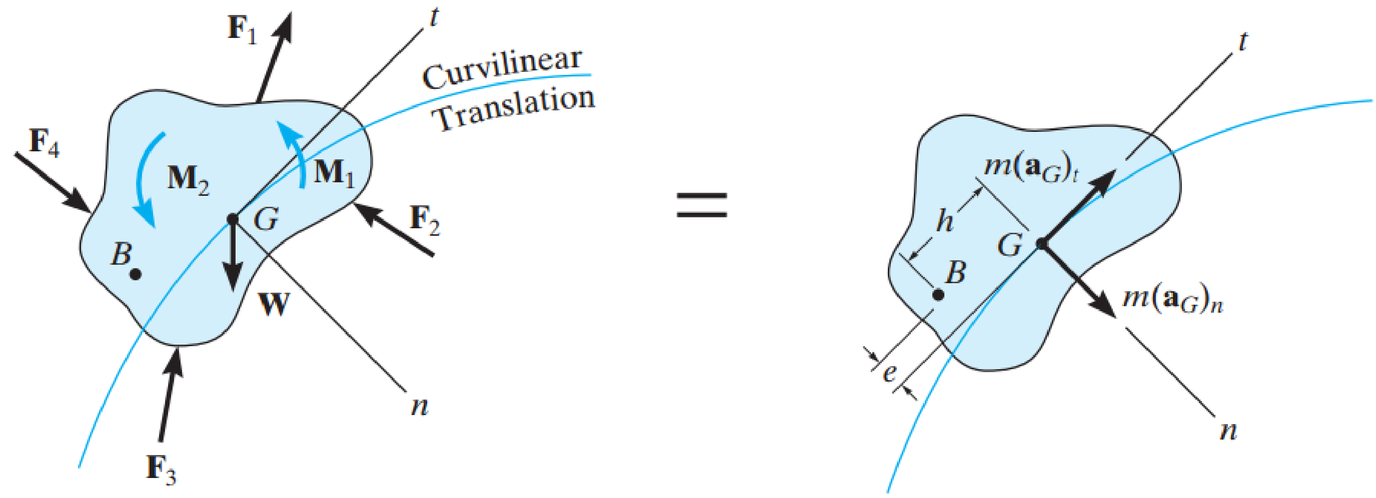
<u>Curvilinear Translation</u>

All the particles of the body have the same accelerations as they travel along curved paths. The three scalar equations of motion are then

$$\Sigma F_n = m(a_G)_n \qquad \Sigma F_t = m(a_G)_t \qquad \Sigma M_G = 0$$

If moments are summed about the arbitrary point B, then it is necessary to account for the moments, $\sum(M_k)_B$ of the two components.

The components of moment are $m(a_G)_n$ and $m(a_G)_t$ about point B.



From the kinetic diagram, h and e represent the perpendicular distances (or “moment arms”) from B to mass center.

The required moment equation therefore becomes

$$\circlearrowleft + \Sigma M_B = \Sigma(\mathcal{M}_k)_B; \qquad \Sigma M_B = e[m(a_G)_t] - h[m(a_G)_n]$$

<u>Rotation about a Fixed Axis</u>

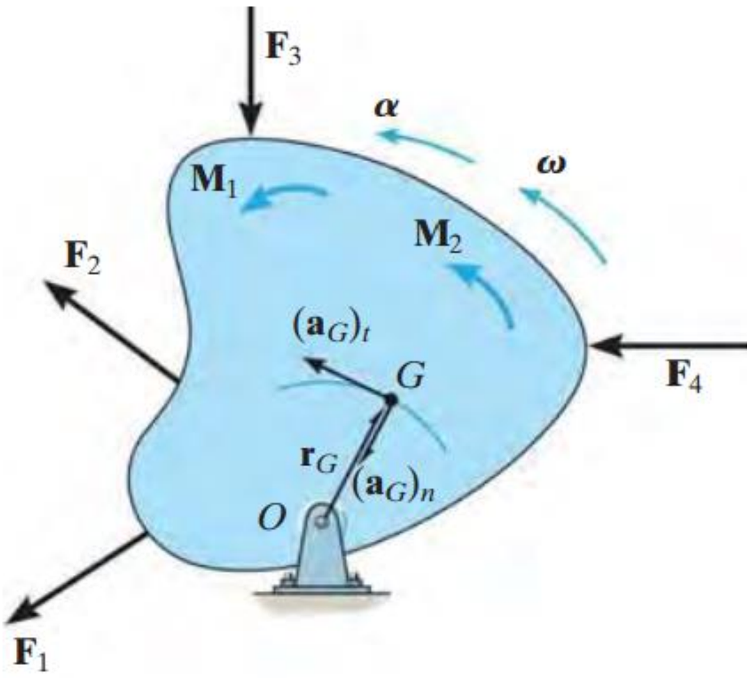
Consider a rigid body (or slab) which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at O.

The angular velocity and angular acceleration are caused by the external force and couple moment system acting on the body.

Because the body’s center of mass G moves around a circular path, the acceleration of this point can be represented by its tangential and normal components.

The tangential component of acceleration of the body’s mass center has a magnitude of $(a_G)_t = \alpha r_G$.

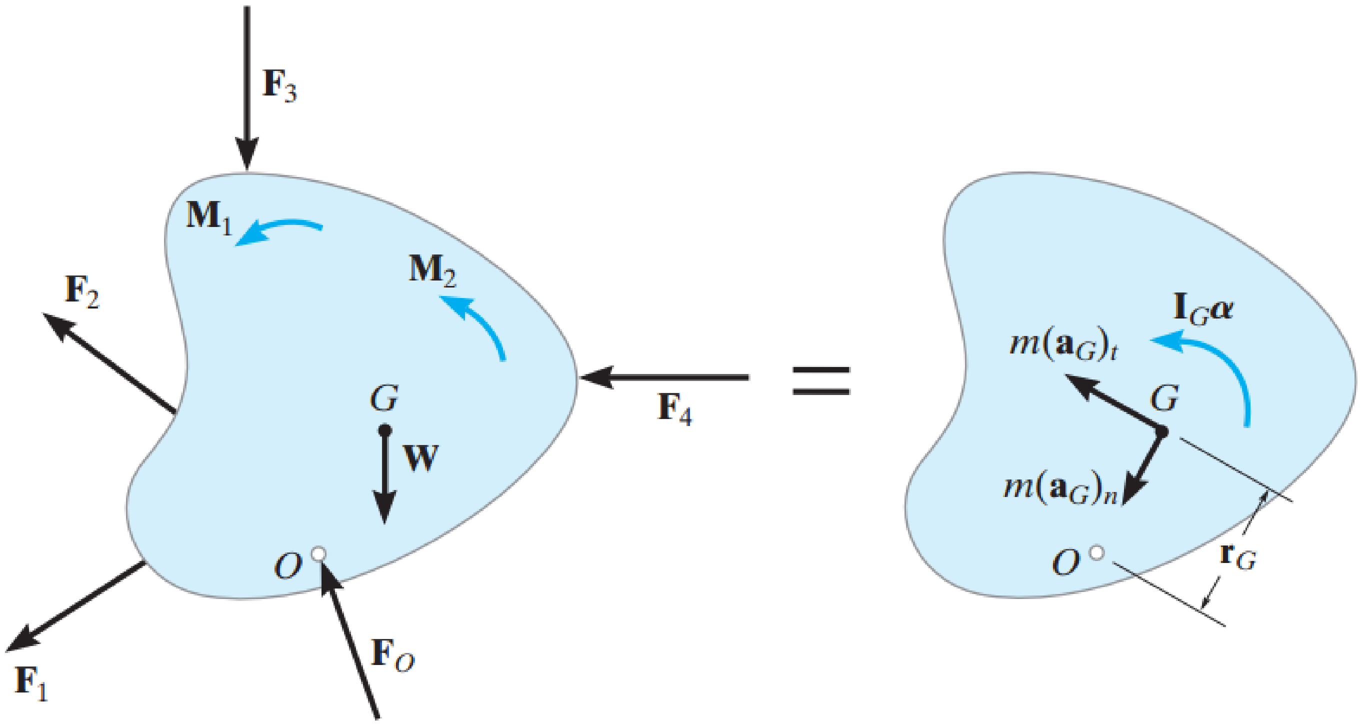
The magnitude of the normal component of acceleration is $(a_G)_n = \omega^2 r_G$. $(a_G)_n$ component is always directed from point G to O, regardless of the rotational sense of ω.



.

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G \qquad \Sigma F_t = m(a_G)_t = m\alpha r_G$$

$$\Sigma M_G = I_G \alpha$$



I_G is the body's moment of inertia calculated about an axis which is perpendicular to the page and passes through G.

Often it is convenient to sum moments about the pin at O in order to eliminate the unknown force F_O.

$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad \Sigma M_O = r_G m(a_G)_t + I_G \alpha$$

Note that the moment of $m(a_G)_n$ is not included here since the line of action of this vector passes through O.

Substituting $(a_G)_t = r_G\alpha$, we may rewrite the above equation as

$\Sigma M_O = (I_G + m r_G^2)\alpha$.

$I_G + m r_G^2$ represent the moment of inertia of the body about the fixed axis of rotation passing through O (I_O). $I_O\alpha$ accounts for the "moment" of both: $m(a_G)_t$ and I_G α about point O

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G \qquad \Sigma F_t = m(a_G)_t = m\alpha r_G$$

$$\Sigma M_O = I_O \alpha$$

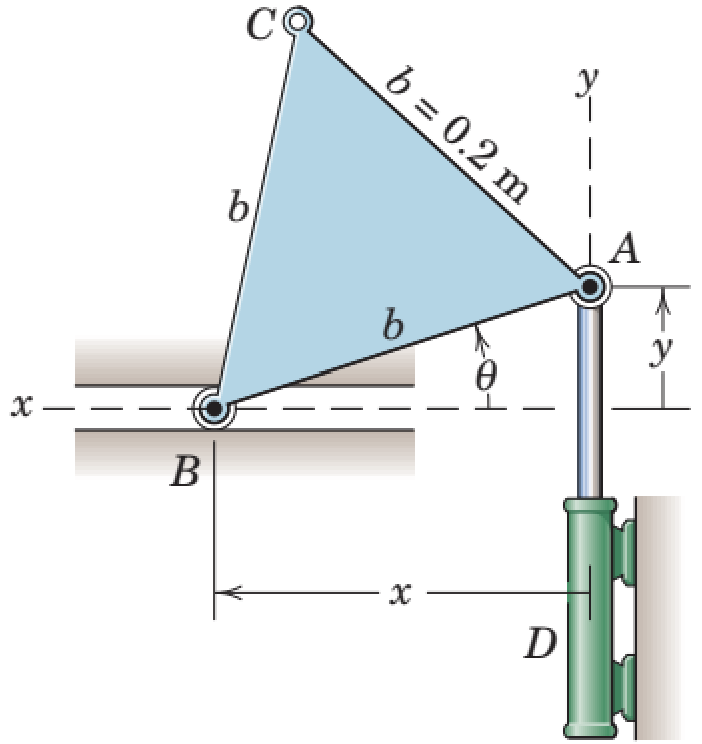
<u>General Planar Motion</u>

The three equations of motion for the general planar motion are similar to case of rotation.

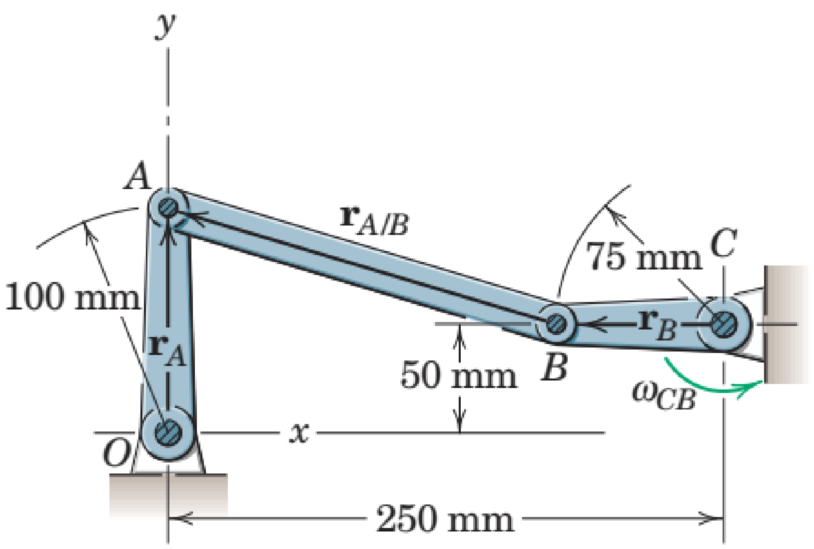
$$\Sigma F_x = m(a_G)_x \qquad \Sigma M_G = I_G \alpha$$
$$\Sigma F_y = m(a_G)_y \qquad \Sigma M_P = \Sigma(\mathcal{M}_k)_P$$

$\Sigma(M_k)_P$ represents the moment sum of $I_G\alpha$ and $m\mathbf{a}_G$.

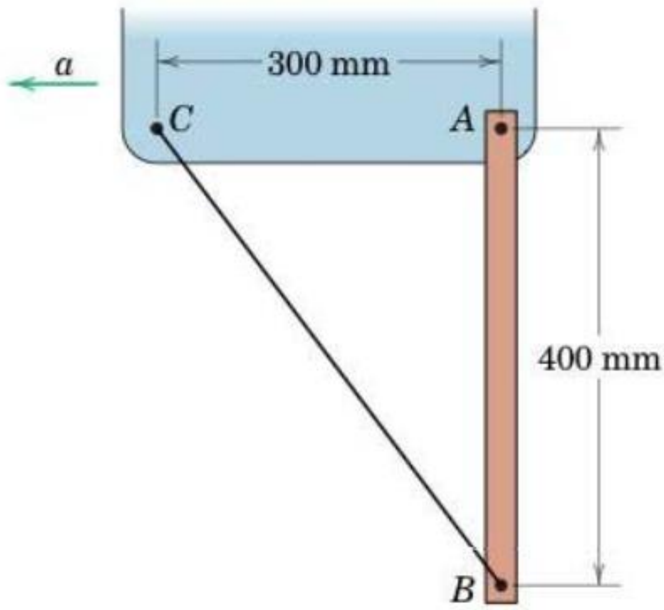
Motion of the equilateral triangular plate ABC in its plane is controlled by the hydraulic cylinder D. If the piston rod in the cylinder is moving upward at the constant rate of 0.3 m /s during an interval of its motion, calculate for the instant when = 30° the velocity and acceleration of the center of the roller B in the horizontal guide and the angular velocity and angular acceleration of edge CB.



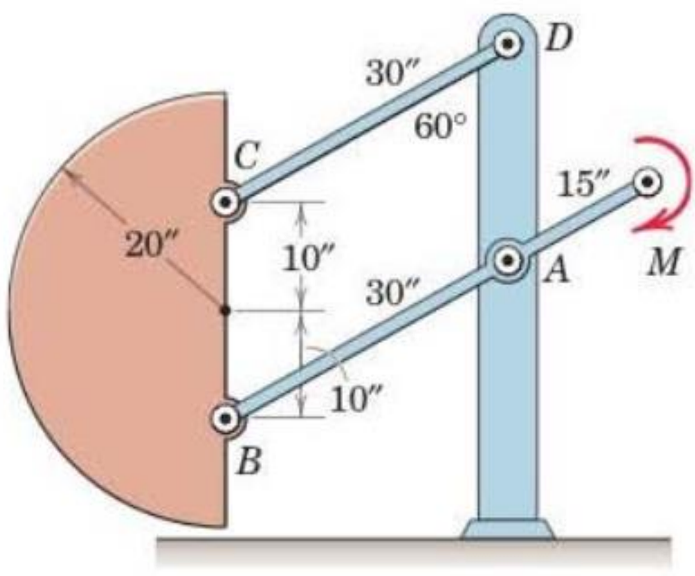
Crank CB oscillates about C through a limited arc, causing crank OA to oscillate about O. When the linkage passes the position shown with CB horizontal and OA vertical, the angular velocity of CB is 2 rad /s counterclockwise. For this instant, determine the angular velocities of OA and AB



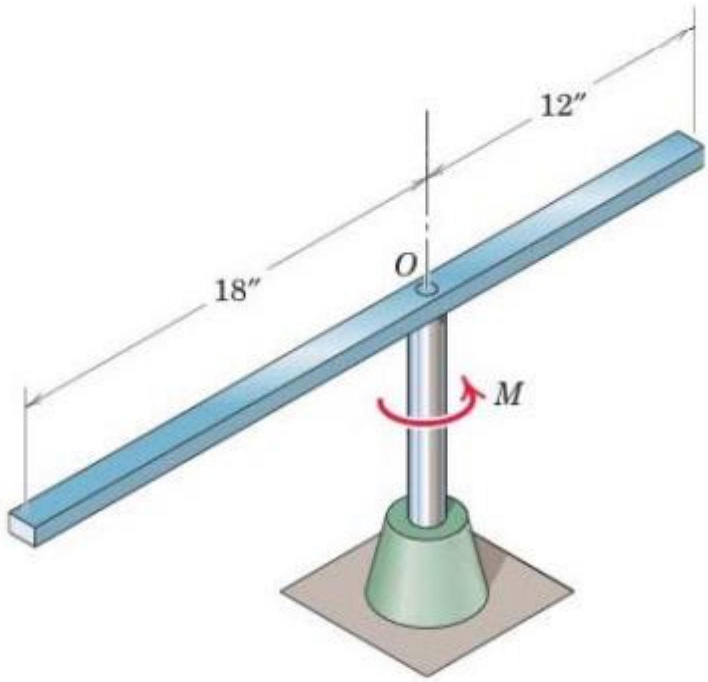
The uniform 5 kg bar is suspended in a vertical position with an accelerating object and restrained by wire BC. If the acceleration is a = 0.6g, determine tension T in the wire and magnitude of total force supported by pin at A.



A semicircular plate of uniform thickness weighs 150 lb and is raised from rest by the parallel linkage of negligible weight under the action of a 500 lb.ft couple M applied at the end of the link. Calculate the components normal and tangent to AB of the shear force supported by the pin at A an instant after the couple M is applied.



The 30-in slender bar weighs 20 lb and is mounted on a vertical shaft at O. If a torque M = 100 lb.in is applied to the bar through its shaft, calculate the horizontal force R on the bearing as the bar starts to rotate.



The slender bar weighs 60 lb and moves in the vertical plane, with its ends constrained to follow the smooth horizontal and vertical guides. If the 30-lb force is applied at A with the bar initially at rest in the position for which angle is 30°, calculate the resulting angular acceleration and the forces on the small end rollers at A and B.